Computers and Mathematics with Applications 58 (2009) 558–565

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Transport–diffusion models with nonlinear boundary conditions and solution by generalized collocation methods

E. De Angelis [a,*], R. Revelli [b], L. Ridolfi [b]

[a] Department of Mathematics, Politecnico di Torino, Torino, Italy

[b] Department of Hydraulics, Politecnico di Torino, Torino, Italy

ARTICLE INFO

Article history:
Received 15 October 2008
Accepted 2 February 2009

Keywords:
Collocation methods
Nonlinear problems
Transport
Diffusion
Macroscopic from microscopic

ABSTRACT

This paper deals with the derivation of a class of nonlinear transport and diffusion models implemented with nonlinear boundary conditions. Mathematical tools to treat the initial-boundary value problems are developed, based on generalized collocation methods, focused on the treatment of nonlinear boundary conditions in one space dimension. Applications refer to a problem of interest in applied sciences.

1. Introduction

Nonlinear transport–diffusion models arise in several fields of applied sciences as documented in classical books devoted to modelling, mathematical methods and applications [1–3]. Some interesting engineering applications motivate the study of initial-boundary value problems with nonlinear boundary conditions as documented in traffic flow modelling [4,5]. In fact, the measurement of the boundary conditions cannot be done directly on the dependent variables, but on suitable nonlinear functions or functionals involving these variables. The computational treatment of the above mathematical problems involves remarkable technical difficulties, which need the development of appropriate mathematical tools.

This paper deals both with the derivation of nonlinear transport–diffusion models and with the computational treatment of initial-boundary value problems with nonlinear boundary conditions. The solution approach is based on a suitable development of generalized collocation methods, that already in [4] has shown to be able to provide reliable simulations.

The generalized collocation method, originally called the ***differential quadrature method***, is a well-known technique to solve numerically initial-boundary problems described by nonlinear partial differential equations. The key feature of the method is to transform the original continuous model (and problem) into a discrete (in space) and continuous (in time) model, with a finite number of degrees of freedom. In this way, the initial-boundary value problem is transformed into an initial value problem for ordinary differential equations simply, where the boundary conditions are implemented in the discretization points corresponding to the boundary points of the collocation.

Several works in applied mathematics demonstrate interest in this method. It was first proposed by Bellman and Casti [6] and then developed by several authors in deterministic [7] and stochastic frameworks [8]. Additional applications and developments are due, among others, to Chen [9,10] in the context of ordinary and partial differential equations, Shu and Richards [11] concerning the solution of Navier–Stokes equations (see also the valuable monograph Shu [11] dealing with various aspects of the generalized quadrature method in the context of fluid mechanics), Bert and Malick [12,13], on the

* Corresponding author.
E-mail addresses: elena.deangelis@polito.it (E. De Angelis), roberto.revelli@polito.it (R. Revelli), luca.ridolfi@polito.it (L. Ridolfi).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.02.034

analysis of the vibration of cylinder shells or rectangular plates, Karami and Malekzadeh [14,15], concerning the application of the method to various problems of vibration analysis, and by Artioli, Gould and Viola [16] and Artioli and Viola [17], on the solution of mathematical problems in structural mechanics.

It is well documented that the method can provide a useful discretization of continuous models and efficiently deals with nonlinearities including the ones related to implicit boundary conditions. This explains the strong interest by engineers, as is documented in the review paper by Bert and Malik [12], where an interesting and detailed report on the application of the original differential quadrature (collocation) method to several engineering problems is provided.

On the other hand, it is known that the method does not generally work in some circumstances. For instance, referring to the Dirichlet problem, the classical Lagrange interpolation is not useful to deal with problems in unbounded domains or with solutions that are oscillating, with high frequency, in the space variables. This problem can be overcome by a suitable use of Sinc functions in the analysis of nonlinear problems; see Lund and Bowers [18], and Stenger [19,20]. The use of Sinc functions was introduced by Bellomo and Ridolfi [21] and subsequently applied to the solution of various problems in applied sciences; see [22–25].

In general, one can conclude that several efforts have been devoted to generalize and improve the mathematical method, which has subsequently been applied to many engineering problems. These improvements have generated a mathematical method, called the ***generalized collocation method***, which is useful to solve a large class of nonlinear problems in applied sciences. The recent book by Bellomo et al. [26] describes thoroughly the various generalizations of the method with applications to the solutions of several problems of interest in engineering sciences.

In this picture, the present paper develops the application of the method to one-dimensional nonlinear initial-boundary value problems where nonlinear boundary conditions are set. This class of mathematical problems entails generally remarkable numerical difficulties. This explains why nonlinear boundary conditions are not frequent in scientific literature, where several problems are simplified by more or less arbitrarily neglecting the nonlinearities in the boundary conditions rather than taking them into account.

The content of this paper is in two parts. The first one, namely Section 2, is devoted to modelling and derivation of nonlinear diffusion equations and to the statement of problems with nonlinear boundary conditions. The second part is devoted to the development of the mathematical method, Section 3, and to some sample simulations, Section 4. The last section also analyses some research perspectives.

2. On the derivation of nonlinear diffusion problems

This section deals with the derivation of transport diffusion models in one space dimension for a physical system constituted by self-propelled particles. This type of fluid dynamics can occur in several complex biological systems [27,28], as well as in some engineering systems such as vehicles on roads [5,29].

The natural way to derive macroscopic equations consists in developing suitable asymptotic limits to average the underlying microscopic description delivered by models of the generalized kinetic theory [30,31], or the kinetic theory for active particles [32–34]. As is known, diffusive or hyperbolic limits can be obtained by a suitable selection of the scaling, as also documented in the case of classical particles [35], for a simple fluid or for mixtures [36], or complex fluids [37,38].

It can be shown how transport–diffusion models can be derived by a heuristic closure of the mass conservation equation simply by relating the local mean velocity to local density and density gradients. This simple approach leads to models characterized by a first-order nonlinear transport term and by a second-order nonlinear diffusion term. Subsequently, it is shown that the statement of some initial-boundary value problems of interest in engineering sciences involves nonlinear boundary conditions.

Bearing all this in mind, let as consider a one-dimensional flow in a duct with length ℓ, and denote by $\rho = \rho(t, x)$ the mass density, while V is the mean velocity. The mass conservation equation is written as

$$\partial_t \rho + \partial_x(\rho V) = 0. \tag{2.1}$$

Let us now introduce the dimensionless variables $u = \rho/\rho_M$, $v = V/V_M$ and let us refer time and space to $T_c = \ell/V_M$ and ℓ, respectively, where ρ_M and V_M denote the maximal admissible values of mass density and velocity. Therefore the mass conservation equation is as follows:

$$\partial_t u + \partial_x(uv) = 0, \tag{2.2}$$

where $u = u(t, x)$ and $v = v(t, x)$.

A simple phenomenological model can be stated by assuming that the velocity takes a maximal value for $u = 0$ and decays to zero for increasing values of the density. A simple way to represent the above behavior is as follows:

$$v = v(u) = \eta \left(1 - u^\star(u, \partial_x u)\right), \qquad u^\star = u\left(1 + \varepsilon(1-u)\partial_x u\right), \tag{2.3}$$

where η and ε are positive parameters.

Remark 2.1. $u^\star$ has been called the ***apparent local density*** in traffic flow modelling, where specifically $\eta = 1$. It takes into account the fact that the driver is not able to perceive exactly the local density u, but perceives a density $u^\star$ which is larger than the real one if the local density gradient is positive, while it is smaller than the real one if the gradient is negative. In addition, the above multiplicative effect increases with decreasing density. More generally, local gradients slow down the flow (if positive), or increase the velocity (if negative).

Introducing the model (2.3) into Eq. (2.2) yields

$$\partial_t u + \eta\,(1-2u)\,\partial_x u = \varepsilon\,\eta\,u\,(2-3u)\,(\partial_x u)^2 + \varepsilon\,\eta\,u^2(1-u)\,\partial_{xx} u. \tag{2.4}$$

Remark 2.2. The quantities ρ_M and V_M are identified by reasoning on the physics of the system. In principle, the solution of mathematical problems related to Eq. (2.4) should deliver $u \in [0, 1]$. If not, the model should be critically analysed.

Remark 2.3. The model is such that if $\varepsilon \to 0$ with $\eta > 0$, a nonlinear hyperbolic equation is obtained as follows:

$$\partial_t u + \eta\,(1-2u)\,\partial_x u = 0. \tag{2.5}$$

On the other hand, if $\eta \to 0$ with $\eta\,\varepsilon \to k > 0$, the following nonlinear parabolic equation is obtained:

$$\partial_t u = k\,u\,(2-3u)(\partial_x u)^2 + k\,u^2(1-u)\,\partial_{xx} u. \tag{2.6}$$

It is interesting to notice that parabolic (diffusion) and hyperbolic (transport) models can be obtained, as already mentioned, by the underlying description offered by statistical mechanics. For instance, by adding a random fluctuation to the transport equation

$$(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}})f(t,\mathbf{x},\vec{\xi}) = \nu\,\mathcal{L}(f)(t,\mathbf{x},\vec{\xi}), \tag{2.7}$$

where $\mathbf{x}$ and $\vec{\xi}$ define the position and velocity of individual particles and f is the distribution function describing the overall state of the system, the linear transport term describes the dynamics of biological organisms modelled by a velocity-jump process,

$$\mathcal{L}(f) = \int_V \left[T(\vec{\xi},\vec{\xi}^*)f(t,\mathbf{x},\vec{\xi}^*,u) - T(\vec{\xi}^*,\vec{\xi})f(t,\mathbf{x},\vec{\xi})\right]\mathrm{d}\vec{\xi}^*, \tag{2.8}$$

where ν is the turning rate or turning frequency and $T(\vec{\xi},\vec{\xi}^*)$ is the probability kernel for the new velocity $\vec{\xi} \in V$ assuming that the previous velocity was $\vec{\xi}^*$.

Parabolic and hyperbolic equations are obtained by different scaling. Respectively,

$$\left(\partial_t + \vec{\xi}\cdot\nabla_{\mathbf{x}}\right)f_\epsilon = \frac{1}{\epsilon}\mathcal{L}(f_\epsilon), \qquad \nu = \frac{1}{\epsilon}, \tag{2.9}$$

and

$$\epsilon\,\partial_t f_\epsilon + \vec{\xi}\cdot\nabla_{\mathbf{x}} f_\epsilon = \frac{1}{\epsilon^p}\,\mathcal{L}(f_\epsilon), \qquad \nu = \frac{1}{\epsilon^p}, \tag{2.10}$$

where ϵ is a small dimensionless parameter which tends to zero in the hydrodynamic limit.

Macroscopic equations are obtained by studying the convergence of f_ϵ to f and by averaging to obtain macroscopic variables. The above approach has the advantage that, by additional work documented in [33,34], transport and diffusion equations can also be obtained in the case of fluids with self-propelled particles, that generate additional source terms.

The above considerations are discussed in the last section of this paper, while we now simply focus on Eq. (2.4) and, specifically, on the statement of initial-boundary value problems, which are obtained by implementing the initial conditions $u_0(x) = u(0,x)$ and boundary conditions at $x = 0$ and $x = 1$.

This statement has to tackle the technical difficulty of measuring u, or $\partial_x u$, at the boundaries. In fact, in several engineering problems the measurement of the flow $q = uv$ is more accurate with respect to that of the density. Therefore, when this is the case and the flow can be measured at $x = 0$ and $x = 1$, the statement of the boundary conditions is as follows:

$$\begin{cases} \psi(t) = q(t,0) = h(\alpha(t),\gamma(t)) = \eta\,\alpha(t)\;[1-\alpha(t)][1-\varepsilon\,\alpha(t)\gamma(t)]\,, \\ \mu(t) = q(t,1) = k(\beta(t),\delta(t)) = \eta\,\beta(t)\;[1-\beta(t)][1-\varepsilon\,\beta(t)\delta(t)]\,, \end{cases} \tag{2.11}$$

where

$$\alpha(t) = u(t,0), \qquad \beta(t) = u(t,1), \qquad \gamma(t) = \partial_x u(t,0), \qquad \delta(t) = \partial_x u(t,1). \tag{2.12}$$

Namely, the boundary conditions are defined by a nonlinear combination of the classical Dirichlet and Neumann boundary conditions. The boundary conditions at $t = 0$ must be consistent with the initial conditions at $x = 0$ and $x = 1$:

$$\begin{cases} \psi(t=0) = h\,(\alpha(t=0),\gamma(t=0)) = h\left(u_0(x=0),\,\dfrac{\mathrm{d}u_0}{\mathrm{d}x}(x=0)\right), \\[2ex] \mu(t=0) = h\,(\beta(t=0),\delta(t=0)) = h\left(u_0(x=1),\,\dfrac{\mathrm{d}u_0}{\mathrm{d}x}(x=1)\right). \end{cases} \tag{2.13}$$

The mathematical method developed in Section 3 shows how the generalized collocation method can provide accurate solutions to the initial-boundary value problems that have been defined above.

3. Solution method and simulations

This section deals with the development of a solution method for initial-boundary value problems for the transport and diffusion models described in the preceding section, namely Eq. (2.4) with initial conditions $u_0(x) = u(0, x)$ and boundary conditions (2.11). As already mentioned, the solution technique is based on a suitable development of the so-called generalized collocation method [26]. As we shall see, this method appears to be particularly suited to deal with nonlinear time-dependent boundary conditions. Let us clarify that there are some other different techniques for transport–diffusion equations that can be used, for instance finite volumes [39,40], or finite differences [41], but the corresponding treatment of the nonlinear boundary conditions can be a stiff task.

The first step consists in deriving, by a space discretization method followed by an interpolation of the dependent variable, a finite model in terms of ordinary differential equations, corresponding to the continuous model (2.4).

Bearing this in mind, let us consider the discretization of the space domain by the collocation

$$I_x = \{x_1 = 0, \ \dots, x_i, \ \dots, \ x_n = 1\}, \tag{3.1}$$

where Chebychev-type collocations, with decreasing values of the distance of the collocation points towards the boundary, are generally used.

The dependent variable $u = u(t, x)$ is subsequently interpolated and approximated by the values $u_i(t) = u(t, x_i)$ as follows:

$$u(t, x) \simeq u^n(t, x) = \sum_{j=1}^{n} L_j(x) u_j(t), \tag{3.2}$$

where the terms $L_j(x)$ denote the Lagrange polynomials:

$$L_j(x) = \frac{(x - x_1) \dots (x - x_{j-1})(x - x_{j+1}) \dots (x - x_n)}{(x_j - x_1) \dots (x_j - x_{j-1})(x_j - x_{j+1}) \dots (x_j - x_n)}. \tag{3.3}$$

Interpolation (3.2) is used to approximate the spatial partial derivatives in the nodal points of I_x:

$$\partial_x u(t; x_i) \simeq \sum_{j=1}^{n} a_{ji}^{(1)} u_j(t) \quad \text{and} \quad \partial_{xx} u(t; x_i) \simeq \sum_{j=1}^{n} a_{ji}^{(2)} u_j(t), \tag{3.4}$$

where

$$a_{ji}^{(1)} = \frac{\mathrm{d}L_j}{\mathrm{d}x}(x_i) \quad \text{and} \quad a_{ji}^{(2)} = \frac{\mathrm{d}^2 L_j}{\mathrm{d}x^2}(x_i), \tag{3.5}$$

and where the detailed expressions of the coefficients $a_{ji}^{(1)}$ are as follows:

$$a_{ji}^{(1)} = \frac{\prod(x_i)}{(x_i - x_j)\prod(x_j)}, \quad \text{for } j \neq i, \tag{3.6}$$

and

$$a_{ii}^{(1)} = \sum_{k \neq i} \frac{1}{x_i - x_k}, \quad \text{for } j = i, \tag{3.7}$$

where

$$\prod(x_i) = \prod_{p \neq i} (x_i - x_p), \qquad \prod(x_j) = \prod_{p \neq j} (x_j - x_p), \tag{3.8}$$

while similarly the second-order coefficients $a_{ji}^{(2)}$ may be computed by a recurrence formula.

Replacing the above expressions (3.2)–(3.8) in Eq. (2.4) for $i = 1, \dots, n$, leads to the discretization of the continuous model (2.4) as follows:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \eta\,(2u_i - 1) \sum_{j=1}^{n} a_{ji}^{(1)} u_j(t) + \varepsilon\,\eta\,u_i(2 - 3\,u_i) \left(\sum_{j=1}^{n} a_{ji}^{(1)} u_j(t)\right)^2 + \varepsilon\,\eta\,u_i^2(1 - u_i) \sum_{j=1}^{n} a_{ji}^{(2)} u_j(t), \tag{3.9}$$

for $i = 1, \dots, n$.

The mathematical problem consists now in inserting the boundary conditions given by Eq. (2.11) into the discrete system (3.9). Recalling that $\alpha(t) \equiv u_1(t)$ and $\beta(t) \equiv u_n(t)$, let us derive ψ and μ with respect to time. The result is as follows:

$$\begin{cases} \psi'(t) = \partial_\alpha h(\alpha, \gamma)\alpha'(t) + \partial_\gamma h(\alpha, \gamma)\gamma'(t), \\ \xi'(t) = \partial_\beta k(\beta, \delta)\beta'(t) + \partial_\delta k(\beta, \delta)\delta'(t), \end{cases} \tag{3.10}$$

where we denote $f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t)$. The time derivatives ψ' and μ' of ψ and μ, respectively, are given as known functions of time.

The terms γ and δ, which appear in Eq. (3.10), can be approximated by the following Lagrange interpolation:

$$\gamma(t) = a_{11}^{(1)}\alpha(t) + \sum_{j=2}^{n-1} a_{j1}^{(1)} u_j(t) + a_{n1}^{(1)}\beta(t), \tag{3.11}$$

and

$$\delta(t) = a_{1n}^{(1)}\alpha(t) + \sum_{j=2}^{n-1} a_{jn}^{(1)} u_j(t) + a_{nn}^{(1)}\beta(t), \tag{3.12}$$

and the derivation with respect to time yields

$$\begin{cases} \gamma'(t) = a_{11}^{(1)}\alpha'(t) + \sum_{j=2}^{n-1} a_{j1}^{(1)} u_j'(t) + a_{n1}^{(1)}\beta'(t), \\ \delta'(t) = a_{1n}^{(1)}\alpha'(t) + \sum_{j=2}^{n-1} a_{jn}^{(1)} u_j'(t) + a_{nn}^{(1)}\beta'(t)\cdot \end{cases} \tag{3.13}$$

Substituting Eq. (3.13) into Eq. (3.10) yields

$$\begin{cases} \left[\partial_\alpha h(\alpha,\gamma) + a_{11}^{(1)}\,\partial_\gamma h(\alpha,\gamma)\right]\alpha'(t) + \left[a_{n1}^{(1)}\,\partial_\gamma h(\alpha,\gamma)\right]\beta'(t) = \psi'(t) - \partial_\gamma h(\alpha,\gamma)\sum_{j=2}^{n-1} a_{j1}^{(1)} u_j', \\ \left[a_{1n}^{(1)}\,\partial_\delta h(\beta,\delta)\right]\alpha'(t) + \left[\partial_\beta h(\beta,\delta) + a_{nn}^{(1)}\,\partial_\delta h(\beta,\delta)\right]\beta'(t) = \mu'(t) - \partial_\delta h(\beta,\delta)\sum_{j=2}^{n-1} a_{jn}^{(1)} u_j'. \end{cases} \tag{3.14}$$

System (3.14) can be algebraically solved to respect $\alpha'(t)$ and $\beta'(t)$, and what we get replaces the first and the last equations in system (3.9). Therefore, the solution of the initial-boundary value problem is obtained by the numerical solution of the initial-value problem for ordinary differential equations. Subsequently, the continuous interpolation of the solution is obtained by Eq. (3.2).

Of course, the above procedures drastically simplifies when only one of the boundary conditions is nonlinear, for instance $\psi(t) = h(u_1(t), \gamma(t))$, while $u_n(t)$ is given; or $\mu(t) = k(\beta(t), \delta(t))$ while $u_1(t)$ is given. Technical calculations are left to the interested reader.

4. Simulations and critical analysis

In this section we apply the solution method presented in the preceding section to sample initial-boundary value problems for the transport diffusion model (2.4). It is worth recalling that such a model has been used, with $\eta = 1$ and $\varepsilon > 0$, to describe vehicular traffic flow [5]. In this case, effectively the measurements of flow are more accurate than those of vehicle density [42]. The same physical situation occurs for crowd dynamics in two space dimensions [43–45].

Specifically, the problem is studied for the initial conditions

$$u(t=0,x) = u_0(x) = 4x^2(x-1)^2, \tag{4.1}$$

and boundary conditions corresponding to constantly null inlet flow and time-dependent outlet flow, as follows:

$$\psi(t) = 0, \qquad \mu(t) = \frac{\eta\, t}{100}\exp(1-t). \tag{4.2}$$

The numerical simulations, performed using the technique proposed in Section 3, have taken advantage of the software Mathematica®. The architecture allows one to solve the algebraic system rapidly to obtain the time evolution equation for the variables $\alpha(t) \equiv u_1(t)$ and $\beta(t) \equiv u_n(t)$, which can be plugged into Eq. (3.9), which is solved by standard methods for ordinary differential equations.

The selection of the number of nodes has been made experimentally by *a posteriori* observation of the stability and accuracy of the solution. The use of Mathematica® optimizes the selection of the time step for the numerical integration.

The results of the simulations are shown in Figs. 1–4, which show some numerical results obtained for different combinations of η and ϵ. Two features clearly emerge. Firstly, all simulations appear smooth and regular. No artificial oscillations induced by the numerical scheme occur in any portion of the domain: either where the solution has a wave-like shape or where it is flat. Only in the case shown in Fig. 3 are very weak spurious fluctuations visible near the boundary at $x = 1$, but in any case the right solution is clearly identifiable. The second feature is the small (with respect to the numerical difficulty of the problem) number of nodes used. This testifies the robustness of the proposed method and entails very short

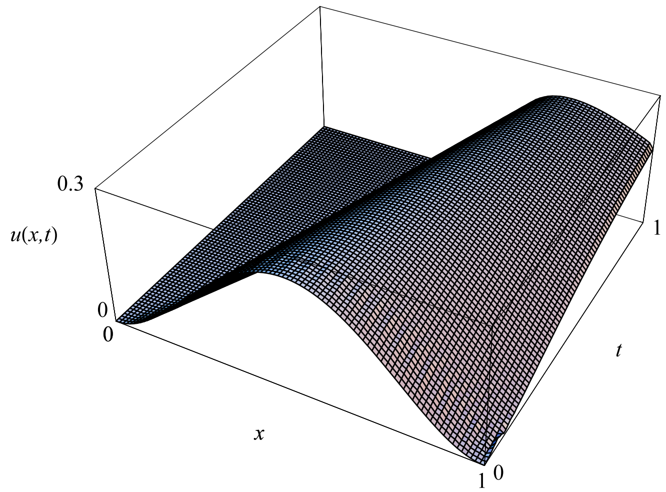

Fig. 1. u versus time and space for $\eta = 0.5$, $\epsilon = 0.1$ and $n = 81$.

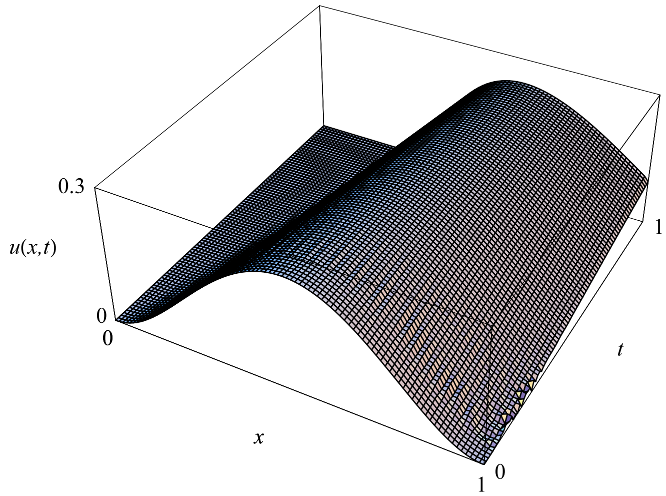

Fig. 2. u versus time and space for $\eta = 0.25$, $\epsilon = 0.1$ and $n = 111$.

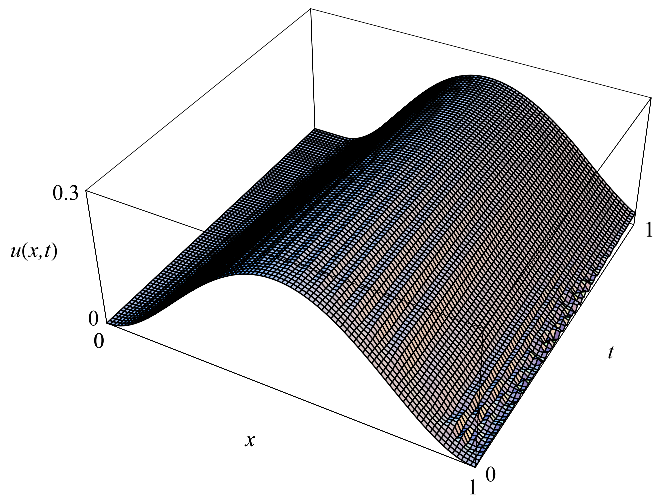

Fig. 3. u versus time and space for $\eta = 0.1$, $\epsilon = 0.1$ and $n = 201$.

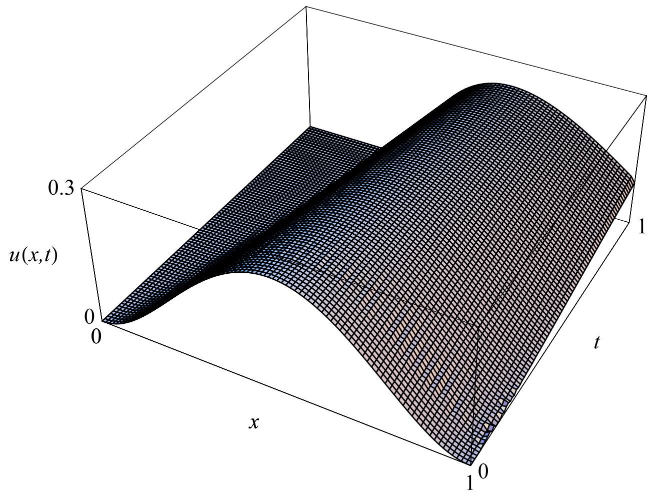

Fig. 4. u versus time and space for $\eta = 0.25$, $\epsilon = 0.2$ and $n = 111$.

computational times. Therefore, the results of the numerical simulations show the capability of the generalized collocation methods to tackle the difficulties due to nonlinear boundary conditions very well, although different methods, as already mentioned, can be tested.

An interesting perspective was already mentioned in Section 2, where the derivation of models with source terms has been outlined for fluids of self-propelled particles in biological fluids or growing tissues [28]. Also, in this specific case, the measurement of flow appears to be relatively more accurate, while the dependent variable is the density. Therefore, the method proposed in this paper can be profitably used after having tackled the technical difficulties induced by the source terms already considered in previous papers [22,25].

References

[1] R. Dautray, J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vols. 1–9, Springer-Verlag, Berlin, 1990.
[2] N. Bellomo, L. Preziosi, Modelling, Mathematical Methods and Scientific Computation, CRC Press, Boca Raton, 1996.
[3] A.C. Fowler, Mathematical Models in the Applied Sciences, in: Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1997.
[4] N. Bellomo, E. De Angelis, L. Graziano, A. Romano, Solution of nonlinear problems in applied sciences by generalized collocation methods and Mathematica, Comput. Math. Appl. 41 (2001) 1343–1363.
[5] N. Bellomo, A. Marasco, A. Romano, From the modelling of driver's behavior to hydrodynamic models and problems of traffic flow, Nonlinear Anal. RWA 3 (2002) 339–363.
[6] R. Bellman, J. Casti, Differential quadrature and long term integration, J. Math. Anal. Appl. 34 (1971) 235–238.
[7] A. Satofuka, A new explicit method for the solution of parabolic differential equation, in: Numerical Methodologies and Properties in Heat Transfer, Hemisphere, New York, 1983, pp. 97–108.
[8] N. Bellomo, F. Flandoli, Stochastic partial-differential equations in continuum physics-on the foundations of the stochastic interpolation method for Ito's type equations, Math. Comput. Simul. 31 (1–2) (1989) 3–17.
[9] C.N. Chen, Generalization of differential quadrature discretization, Numer. Algorithms 22 (1999) 167–182.
[10] C.N. Chen, Differential quadrature element analysis using extended differential quadrature, Comput. Math. Appl. 39 (2000) 65–79.
[11] C. Shu, B.E. Richards, Application of generalized differential quadrature to solve two-dimensional incompressible Navier–Stokes equations, Internat. J. Numer. Methods Fluids 15 (1992) 791–798.
[12] C. Bert, M. Malik, Differential quadrature method in computational mechanics, ASME Rev. 49 (1996) 1–27.
[13] C. Bert, M. Malik, Three-dimensional elasticity solutions for free vibrations of rectangular plates by the differential quadrature method, Internat. J. Solids Struct. 35 (1998) 299–318.
[14] G. Karami, P. Malekzadeh, A new differential quadrature methodology for beam analysis and the associated differential quadrature element method, Comput. Methods Appl. Mech. Engrg. 191 (2002) 3509–3526.
[15] G. Karami, P. Malekzadeh, Vibration of non-uniform thick plates on elastic foundation by differential quadrature method, Eng. Struct. 26 (2004) 1473–1482.
[16] E. Artioli, P. Gould, E. Viola, A differential quadrature method solution for the shear-deformable shells of revolution, Eng. Struct. 27 (2005) 1879–1892.
[17] E. Artioli, E. Viola, Static analysis of shear-deformable shells of revolution via G.D.Q. method, Struct. Eng. Mech. 19 (2005) 459–475.
[18] J. Lund, K. Bowers, Sinc Methods, SIAM, Philadelphia, 1992.
[19] F. Stenger, Numerical methods based on Whittaker cardinal or sinc functions, SIAM Rev. 23 (1983) 165–224.
[20] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer, Berlin, 1993.
[21] N. Bellomo, L. Ridolfi, Solution of nonlinear initial-boundary value problems by sinc collocation-interpolation methods, Comput. Math. Appl. 29 (4) (1995) 15–28.
[22] R. Revelli, L. Ridolfi, Sinc collocation-interpolation method for the simulation of nonlinear waves, Comput. Math. Appl. 46 (8-9) (2003) 1443–1453.
[23] R. Revelli, L. Ridolfi, P. Massarotti, Nonlinear convection–dispersion models with a distributed pollutant source. I. Direct initial boundary value problems, Math. Comput. Modelling 39 (9–10) (2004) 1023–1034.
[24] N. Bellomo, Nonlinear models and problems in applied sciences: From differential quadrature to generalized collocation methods, Math. Comput. Modelling 26 (4) (1997) 13–34.
[25] R. Revelli, L. Ridolfi, Nonlinear convection–dispersion models with a localized pollutant source. II. A class of inverse problems, Math. Comput. Modelling 42 (2005) 601–612.
[26] N. Bellomo, B. Lods, R. Revelli, L. Ridolfi, Generalized Collocation Methods — Solutions to Nonlinear Problems, Birkhäuser, 2008.

[27] N. Bellomo, N.K. Li, P.K. Maini, On the foundations of cancer modelling: Selected topics, speculations and perspectives, Math. Models Methods Appl. Sci. 18 (2008) 593–646.
[28] N. Bellomo, M. Delitala, From the mathematical kinetic, and stochastic game theory to modelling mutations, onset, progression and immune competition of cancer cells, Phys. Life Rev. 5 (2008) 183–206.
[29] M. Delitala, A. Tosin, Mathematical modelling of vehicular traffic: A discrete kinetic theory approach, Math. Models Methods Appl. Sci. 17 (2007) 901–932.
[30] T. Hillen, H. Othmer, The diffusion limit of transport equations derived from velocity–jump processes, SIAM J. Appl. Math. 61 (2000) 751–775.
[31] F. Filbet, P. Laurençot, B. Perthame, Derivation of hyperbolic models for chemosensitive movement, J. Math. Biol. 50 (2005) 189–207.
[32] N. Bellomo, A. Bellouquid, On the onset of non-linearity for diffusion models of binary mixtures of biological materials by asymptotic analysis, Internat. J. Non-Linear Mech. 41 (2006) 281–293.
[33] N. Bellomo, A. Bellouquid, J. Nieto, J.J. Soler, Multicellular growing systems: Hyperbolic limits towards macroscopic description, Math. Models Methods Appl. Sci. 17 (2007) 1675–1693.
[34] N. Bellomo, A. Bellouquid, M.A. Herrero, From microscopic to macroscopic description of multicellular systems and biological growing tissues, Comput. Math. Appl. 53 (2007) 647–663.
[35] T. Goudon, J. Nieto, F. Poupoud, J. Soler, Multidimensional high-field limit of the electrostatic Vlasov–Poisson–Fokker–Plank system, J. Differential Equations 213 (2005) 418–442.
[36] C. Dogbé, Fluid dynamic limits for gas mixture I: Formal derivations, Math. Models Methods Appl. Sci 18 (2008) 1633–1672.
[37] R. Bürger, A. Garcia, M. Kunik, A generalised kinetic model for sedimentation of polydisperse suspensions with a continuous particles size distribution, Math. Models Methods Appl. Sci 18 (2008) 1741–1787.
[38] P. Degond, S. Motsch, Continuum limit of self-driven particles with orientation interaction, Math. Models Methods Appl. Sci. 18 (2008) 1193–1215.
[39] R. Gobbi, S. Palpacelli, R. Spigler, Numerical treatment of a nonlinear transport equation modelling crystal precipitation, Math. Models Methods Appl. Sci. 18 (2008) 1505–1528.
[40] N. Cavallini, V. Caleffi, V. Coscia, Finite volume and WENO scheme in one-dimensional vascular system modelling, Comput. Math. Appl. 56 (2008) 2382–2397.
[41] C. Dogbé, On the numerical solutions of second order macroscopic models of pedestrian flows, Comput. Math. Appl. 56 (2008) 1884–1898.
[42] B. Kerner, The Physics of Traffic, Springer, 2004.
[43] R.L. Hughes, The flow of human crowds, Annual Rev. Fluid Mech. 35 (2003) 169–183.
[44] V. Coscia, C. Canavesio, First-order macroscopic modelling of human crowd dynamics, Math. Models Methods Appl. Sci. 18 (2008) 1217–1247.
[45] N. Bellomo, C. Dogbé, On the modelling crowd dynamics from scaling to hyperbolic macroscopic models, Math. Models Methods Appl. Sci. 18 (Suppl.) (2008) 1317–1345.